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NASH-EQUILIBRIUM IN STOCHASTIC DIFFERENTIAL GAMES

SVETOSLAV D. GAIDOV
Centre of Mathematics, P.O. Box 325, BG-7000 Rousse, Bulgaria

Abstract—The paper deals with N-person nonzero-sum games in which the dynamics is described by Ito stochastic differential equations. Sufficient conditions are found guaranteeing the Nash-equilibrium for the strategies of the players. The optimal strategies are solutions of certain partial initial value problems analogous to the Bellman equation in the theory of dynamic programming. Linear-quadratic games with and without a control dependent noise are studied.

1. INTRODUCTION

There are different approaches to the optimal control of dynamic stochastic systems. One can mention the martingale methods of Davis, *et al.*, which are very interesting from a theoretic point of view. In this paper we shall follow the approach of Fleming and Rishel[4], but to an optimal control in situations of conflicts, i.e. to stochastic differential games. The martingale methods (see [2]) can be also applied to problems similar to those treated below. The approach chosen here makes possible a further application of the approximation and numerical procedures developed by Chernousko and Kolmanovskii[1]. The game aspect of the problem is described as in Vaisbord and Zhukovskii[7]. The examples of linear-quadratic games come from the linear regulator problems [1,4,6]. The results presented here have been announced without any proof and details in [5].

2. FORMALIZATION OF THE GAME

Let us consider the game

$$\Gamma = \langle\{1, \ldots, N\}, \Sigma, \{\mathfrak{U}_1, \ldots, \mathfrak{U}_N\}, \{\mathfrak{T}_1, \ldots, \mathfrak{T}_N\}\rangle. \tag{2.1}$$

Here $\{1, \ldots, N\}$ is the set of the players participating in the game Γ. The evolution of the dynamic system Σ is described by a stochastic differential equation of the type

$$\mathrm{d}\xi(t) = f(t, \xi, u_1, \ldots, u_N)\,\mathrm{d}t + \sigma(t, \xi, u_1, \ldots, u_N)\,\mathrm{d}w(t), \quad t \in [t_0, T], \tag{2.2}$$

with initial condition $\xi(t_0) = \xi_0 \in \mathrm{R}^n$ and where $T > t_0 \geqq 0$. The process $w = \{w(t), t \in [t_0, T]\}$ is a standard m-dimensional Wiener process, defined on some complete probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ and adapted to a family $\mathbf{F} = \{\mathfrak{F}_t, t \in [t_0, T]\}$ of nondecreasing sub-σ-algebras of $\mathfrak{F}$. $\xi \in \mathrm{R}^n$ is the state vector process and $u_i \in U_i \subset \mathrm{R}^{\nu_i}$ is the control of the ith player, $i = 1, \ldots, N$. Now let us make the following assumptions about the functions $f(t, x, u_1, \ldots, u_N)$ and $\sigma(t, x, u_1, \ldots, u_N)$. Suppose

$$f : [t_0, T] \times \mathrm{R}^n \times U_1 \times \cdots \times U_N \longrightarrow \mathrm{R}^n,$$

and

$$\sigma : [t_0, T] \times \mathrm{R}^n \times U_1 \times \cdots \times U_N \longrightarrow \mathrm{R}^n \times \mathrm{R}^m,$$

have continuous partial derivatives and let $C > 0$ be a constant, such that

$$|f(t, 0, \ldots, 0)| + |\sigma(t, 0, \ldots, 0)| \leqq C,$$
$$|f_x| + |\sigma_x| + \sum_{i=1}^{N} (|f_{u_i}| + |\sigma_{u_i}|) \leqq C,$$

where $|\cdot|$ is a general symbol for the norms in the respective spaces.

Each player has perfect observations of the state vector $\xi(t)$ at every moment $t \in [t_0, T]$ and constructs his strategy in the game (2.1) as an admissible feedback control of the following type:

$$u_i(t) = u_i(t, \xi(t)),$$

where

$$u_i(\cdot, \cdot) : [t_0, T] \times \mathrm{R}^n \longrightarrow U_i$$

is a Borel function satisfying the conditions:

(i) There exists a constant $M_i > 0$ such that

$$|u_i(t, x)| \leqq M_i(1 + |x|), \quad \text{for all } (t, x) \in [t_0, T] \times \mathrm{R}^n;$$

(ii) for each bounded set $B \subset \mathrm{R}^n$ and $T^* \in (t_0, T)$, there exists a constant $K_i > 0$ such that for arbitrary $x, y \in B$ and $t \in [t_0, T^*]$,

$$|u_i(t, x) - u_i(t, y)| \leqq K_i|x - y|.$$

Denote by $\mathfrak{U}_i$ the set of strategies of the ith player, $i = 1, \ldots, N$, and

$$\mathfrak{U} = \prod_{i=1}^{N} \mathfrak{U}_i.$$

Let a vector of strategies $u = (u_1, \ldots, u_N) \in \mathfrak{U}$ be called for brevity just a strategy.

The assumptions made above imply the existence and sample path uniqueness of the solution $\xi = \{\xi(t), t \in [t_0, T]\}$ of (2.2) corresponding to the control $u \in \mathfrak{U}$. The process ξ is a Markov process.

Let L_i, ψ_i be continuous functions satisfying the polynomial growth conditions:

$$|L_i(t, x, u_1, \ldots, u_N)| \leqq C_i(1 + |x| + \sum_{i=1}^{N} |u_i|)^k,$$
$$|\psi_i(t, x)| \leqq C_i(1 + |x|)^k,$$

where k, C_i are positive constants. Introduce now the cost-function $\mathfrak{T}_i(u)$ of the ith player:

$$\mathfrak{T}_i(u) = \mathbf{E}_{t_0,\xi_0}\left\{\psi_i(T, \xi(T)) + \int_{t_0}^{T} L_i(t, \xi, u_1, \ldots, u_N)\, dt\right\}, \quad i = 1, \ldots, N.$$

The object of each player in the game (2.1) is to minimize his own cost-function.

3. MAIN RESULT

Let us suppose that all players choose their strategies according to the principle of the stability of the situation, i.e. the unilateral deviation of a player leads in general to a maximization of his own cost-function. Thus we come to the following

DEFINITION

The strategy $u^e \in \mathfrak{U}$ is called a Nash-equilibrium strategy for the game (2.1), if for each $u_i \in \mathfrak{U}_i$,

$$\mathfrak{T}_i(u_1^e, \ldots, u_{i-1}^e, u_i, u_{i+1}^e, \ldots, u_N^e) \geqq \mathfrak{T}_i(u^e), \quad i = 1, \ldots, N.$$

Next we give sufficient conditions guaranteeing that $u^e \in \mathfrak{U}$ is a Nash-equilibrium strategy for the game (2.1). Denote

$$(u^e \| u_i) = (u_1^e, \ldots, u_{i-1}^e, u_i, u_{i+1}^e, \ldots, u_N^e)$$

and

$$G_i(t, x, u^e \| u_i) = W_t^{(i)}(t, x) + \mathcal{A}(u^e \| u_i) W^{(i)}(t, x) + L_i(t, x, u^e \| u_i),$$

for all $t \in [t_0, T]$, $x \in \mathbf{R}^n$. Here $\mathcal{A}(u^e \| u_i)$ is the infinitesimal operator (see [3]) of the process ξ under the strategy $u^e \| u_i$. Now we present a result analogous to that of Varaiya[8,9].

THEOREM

The strategy u^e is a Nash-equilibrium strategy for the game (2.1), if there exist real-valued functions $W^{(i)}(t, x)$ such that for all $t \in [t_0, T]$, $x \in \mathbf{R}^n$ and $i = 1, \ldots, N$, the following conditions jointly hold:

(a) $W^{(i)}$, $W_t^{(i)}$, $W_x^{(i)}$, $W_{xx}^{(i)}$ are continuous;

(b) $G_i(t, x, u^e \| u_i^e) = 0$;

(c) $G_i(t, x, u^e \| u_i) \geqq 0$ for each strategy $u_i \in \mathfrak{U}_i$;

(d) $W^{(i)}(T, x) = \psi_i(T, x)$.

Proof. Let $\xi^e(t)$ and $\xi^{(i)}(t)$, $t \in [t_0, T]$, be the sample paths of the solutions of (2.2) corresponding to the strategies u^e and $u^e \| u_i$, respectively. Write Ito–Dynkin's formula with u^e and $\xi^e(t)$:

$$W^{(i)}(t, x) = \mathbf{E}_{t,x}\left\{ W^{(i)}(T, \xi^e(T)) - \int_t^T [W_\tau^{(i)}(\tau, \xi^e(\tau)) + \mathcal{A}(u^e \| u_i^e) W^{(i)}(\tau, \xi^e(\tau))] \, \mathrm{d}\tau \right\}.$$

This representation in conjunction with (b) and (d) implies

$$W^{(i)}(t, x) = \mathbf{E}_{t,x}\left\{ \psi_i(T, \xi^e(T)) + \int_t^T L_i(\tau, \xi^e, u^e \| u_i^e) \, \mathrm{d}\tau \right\}$$

and hence

$$W^{(i)}(t_0, \xi_0) = \mathbf{E}_{t_0,\xi_0}\left\{ \psi_i(T, \xi^e(T)) + \int_{t_0}^T L_i(t, \xi^e, u^e \| u_i^e) \, \mathrm{d}t \right\}.$$

Now write again Ito–Dynkin's formula, but with $u^e \| u_i$ and $\xi^{(i)}(t)$:

$$W^{(i)}(t, x) = \mathbf{E}_{t,x}\left\{ W^{(i)}(T, \xi^{(i)}(T)) - \int_t^T [W_\tau^{(i)}(\tau, \xi^{(i)}(\tau)) + \mathcal{A}(u^e \| u_i) W^{(i)}(\tau, \xi^{(i)}(\tau))] \, \mathrm{d}\tau \right\}.$$

Taking into account conditions (c) and (d), we get

$$W^{(i)}(t, x) \leqq \mathbf{E}_{t,x}\left\{\psi_i(T, \xi^{(i)}(T)) + \int_t^T L_i(\tau, \xi^{(i)}, u^e\|u_i)\,d\tau\right\},$$

which leads to

$$W^{(i)}(t_0, \xi_0) \leqq \mathbf{E}_{t_0,\xi_0}\left\{\psi_i(T, \xi^{(i)}(T)) + \int_{t_0}^T L_i(t, \xi^{(i)}, u^e\|u_i)\,dt\right\}.$$

Finally, we obtain

$$W^{(i)}(t_0, \xi_0) = \mathfrak{I}_i(u^e) \leqq \mathfrak{I}_i(u^e\|u_i), \quad \text{for each } u_i \in \mathfrak{U}_i, \quad i = 1, \ldots, N.$$

Therefore, u^e is a Nash-equilibrium strategy for the game (2.1).

Remark. Note that $W^{(i)}(t, x)$ is a solution of a dynamic programming equation of the type

$$\min_{u_i} G_i(t, x, u^e\|u_i) = 0,$$

with a boundary condition

$$W^{(i)}(T, x) = \psi_i(T, x),$$

for all $t \in [t_0, T]$, $x \in \mathrm{R}^n$, $i = 1, \ldots, N$.

4. LINEAR-QUADRATIC GAMES WITH A STATE AND CONTROL INDEPENDENT NOISE

Let us consider the game (2.1), where the evolution of the dynamic system Σ is described by the linear stochastic differential equation

$$d\xi(t) = \left[A(t)\xi + \sum_{i=1}^N B_i(t)u_i\right] dt + \sigma(t)\,dw(t), \quad t \in [t_0, T],$$

with initial condition $\xi(t_0) = \xi_0$. Here ξ_0, w, u_i, $i = 1, \ldots, N$, are the same as in Sec. 2. $A(t)$ is an $n \times n$-matrix, $\sigma(t)$ is an $n \times m$-matrix, and $B_i(t)$ are $n \times \nu_i$-matrices, $i = 1, \ldots, N$. Let further prime denote vector or matrix transposition. The cost-function $\mathfrak{I}_i(u)$ of the ith player is the following quadratic functional:

$$\mathfrak{I}_i(u) = \mathbf{E}_{t_0,\xi_0}\left\{\xi'(T)D_i\xi(T) + \int_{t_0}^T \left[\xi'(t)M_i(t)\xi(t) + \sum_{j=1}^N u_j'(t)N_j^{(i)}(t)u_j(t)\right] dt\right\}.$$

Here D_i, $M_i(t)$ and $N_j^{(i)}(t)$, $j = 1, \ldots, N$, are symmetric matrices with dimensions $n \times n$, $n \times n$ and $\nu_i \times \nu_i$, respectively, $i = 1, \ldots, N$.

Now consider the functions

$$G_i(t, x, u) = W_t^{(i)}(t, x) + \mathscr{A}(u)W^{(i)}(t, x) + L_i(t, x, u) = W_t^{(i)}(t, x) + \left[A(t)x + \sum_{j=1}^N B_j(t)u_j\right]'$$

$$\times\, W_x^{(i)}(t, x) + \tfrac{1}{2}\,\mathrm{tr}[a(t)W_{xx}^{(i)}(t, x)] + x'M_i(t)x + \sum_{j=1}^N u_j'N_j^{(i)}(t)u_j, \quad i = 1, \ldots, N,$$

where $a = \sigma\sigma'$. Conditions (b) and (c) from Sec. 3 are equivalent to the relations

$$\min_{u_i} G_i(t, x, u^e \| u_i) = G_i(t, x, u^e \| u_i^e) = 0, \quad i = 1, \ldots, N. \tag{4.1}$$

Hence the Nash-equilibrium strategies u_i^e are solutions of the following equations:

$$\partial G_i / \partial u_i |_{u_i = u_i^e} = B_i'(t) W_x^{(i)}(t, x) + 2N_i^{(i)}(t) u_i^e = 0,$$

i.e.

$$u_i^e = -\tfrac{1}{2} [N_i^{(i)}(t)]^{-1} B_i'(t) W_x^{(i)}(t, x), \quad i = 1, \ldots, N.$$

Further, put u_i^e, $i = 1, \ldots, N$, in (b) and search $W^{(i)}(t, x)$ in the following special form:

$$W^{(i)}(t, x) = x'\theta_i(t)x + r_i(t), \tag{4.2}$$

where $\theta_i(t)$ is a symmetric $n \times n$-matrix with $\theta_i(T) = D_i$ and $r_i(t)$ is a scalar function, $i = 1, \ldots, N$. Then

$$u_i^e = -[N_i^{(i)}(t)]^{-1} B_i'(t) \theta_i(t) x, \quad i = 1, \ldots, N. \tag{4.3}$$

Thus for relations (b), we obtain

$$\begin{aligned} x' \Big\{ & \dot\theta_i(t) + A'(t)\theta_i(t) + \theta_i(t)A(t) + M_i(t) - \sum_{j=1}^{N} \theta_j(t) B_j(t) [N_j^{(j)}(t)]^{-1} B_j'(t) \theta_i(t) - \theta_i(t) \\ & \times \sum_{j=1}^{N} B_j(t) [N_j^{(j)}(t)]^{-1} B_j'(t) \theta_j(t) + \sum_{j=1}^{N} \theta_j(t) B_j(t) [N_j^{(j)}(t)]^{-1} N_j^{(i)}(t) [N_j^{(j)}(t)]^{-1} B_j'(t) \theta_j(t) \Big\} x \\ & + \dot r_i(t) + \operatorname{tr}[a(t)\theta_i(t)] = 0, \quad i = 1, \ldots, N, \end{aligned}$$

and we come to the following.

Proposition

Let in the linear-quadratic game (2.1) the matrices $A(t)$, $B_i(t)$, $\sigma(t)$, $M_i(t)$. $N_j^{(i)}(t)$ be continuous and D_i be constant. Let the matrices D_i and $M_i(t)$ be nonnegative definite and $N_i^{(i)}(t)$ be positive definite. Now,

(i) If the set of matrices $\theta_i(t)$, $i = 1, \ldots, N$, is the solution of the system of matrix Riccati differential equations

$$\begin{aligned} 0 = {} & \dot\theta_i(t) + A'(t)\theta_i(t) + \theta_i(t)A(t) + M_i(t) - \sum_{j=1}^{N} \theta_j(t) B_j(t) [N_j^{(j)}(t)]^{-1} B_j'(t) \theta_i(t) \\ & - \theta_i(t) \sum_{j=1}^{N} B_j(t) [N_j^{(j)}(t)]^{-1} B_j'(t) \theta_j(t) + \sum_{j=1}^{N} \theta_j(t) B_j(t) [N_j^{(j)}(t)]^{-1} N_j^{(i)}(t) [N_j^{(j)}(t)]^{-1} B_j'(t) \theta_j(t), \\ & i = 1, \ldots, N, \end{aligned}$$

with boundary conditions

$$\theta_i(T) = D_i, \; i = 1, \ldots, N,$$

and

$$r_i(t) = \int_t^T \operatorname{tr}[a(\tau)\theta_i(\tau)]\, d\tau,\ i = 1, \ldots, N;$$

(ii) then $W^{(i)}(t, x)$ given by (4.2) are the solutions of the dynamic programming equations (4.1) and u_i^e, $i = 1, \ldots, N$, defined by (4.3) are the Nash-equilibrium strategies for the linear-quadratic game (2.1) considered in this section.

5. LINEAR-QUADRATIC GAMES WITH CONTROLLED DRIFT AND DIFFUSION COEFFICIENTS

Let the evolution of the dynamic system Σ of the game (2.1) be described by the linear stochastic differential equation

$$d\xi(t) = [A(t)\xi + \sum_{i=1}^{N} B_i(t)u_i]\, dt + \sigma(t, \xi, u_1, \ldots, u_N)\, dw(t), \quad t \in [t_0, T],$$

with initial condition $\xi(t_0) = \xi_0 \in \mathrm{R}$. Here $\xi \in \mathrm{R}$ is the state process, $w = \{w(t), t \in [t_0, T]\}$ is an $(N+2)$-dimensional standard Wiener process and $u_i \in U_i \subset \mathrm{R}$ is the control of the ith player, $i = 1, \ldots, N$. Now $\sigma(t, \xi, u_1, \ldots, u_N)$ is an $1 \times (N+2)$-matrix of the form

$$\sigma = (\sigma_0(t)\xi\ \sigma_1(t)u_1 \cdots \sigma_N(t)u_N\ \sigma_{N+1}(t)).$$

Henceforth $A(t)$, $B_i(t)$, $i = 1, \ldots, N$, $\sigma_j(t)$, $j = 0, \ldots, N+1$, are functions taking values in R. The cost-function $\mathfrak{I}_i(u)$ of the ith player is the functional

$$\mathfrak{I}_i(u) = \mathrm{E}_{t_0,\xi_0}\left\{D_i\xi^2(T) + \int_{t_0}^{T} \left[M_i(t)\xi^2(t) + \sum_{j=1}^{N} N_j^{(i)}(t)u_j^2(t)\right] dt\right\}, \quad i = 1, \ldots, N.$$

Here D_i are constants and $M_i(t)$, $N_j^{(i)}(t)$, $j = 1, \ldots, N$, are real-valued functions, $i = 1, \ldots, N$.

Now consider the functions

$$\begin{aligned} G_i(t, x, u) &= W_t^{(i)}(t, x) + \mathscr{A}(u)W^{(i)}(t, x) + L_i(t, x, u) \\ &= \partial W^{(i)}(t, x)/\partial t + \left[A(t)x + \sum_{j=1}^{N} B_j(t)u_j\right] \partial W^{(i)}(t, x)/\partial x + \frac{1}{2}\left[\sigma_0^2(t)x^2\right. \\ &\quad \left. + \sum_{j=1}^{N} \sigma_j^2(t)u_j^2 + \sigma_{N+1}^2(t)\right] \partial^2 W^{(i)}(t, x)/\partial x^2 + M_i(t)x^2 + \sum_{j=1}^{N} N_j^{(i)}(t)u_j^2, \end{aligned}$$

$i = 1, \ldots, N$. Conditions (b) and (c) imply the relations

$$\min_{u_i} G_i(t, x, u^e\|u_i) = G_i(t, x, u^e\|u_i^e) = 0, \quad i = 1, \ldots, N. \tag{5.1}$$

Hence the Nash-equilibrium strategies u_i^e are the solutions of the equations

$$\partial G_i/\partial u_i|_{u_i=u_i^e} = B_i(t)\, \partial W^{(i)}(t, x)/\partial x + \sigma_i^2(t)\, \partial^2 W^{(i)}(t, x)/\partial x^2 u_i^e + 2N_i^{(i)}(t)u_i^e = 0,$$

i.e.

$$u_i^e = -[\sigma_i^2(t)\, \partial^2 W^{(i)}(t, x)/\partial x^2 + 2N_i^{(i)}(t)]^{-1}B_i(t)\, \partial W^{(i)}(t, x)/\partial x, \quad i = 1, \ldots, N.$$

Next put u_i^e, $i = 1, \ldots, N$, in (b) and search $W^{(i)}(t, x)$ in the following special form

$$W^{(i)}(t, x) = \theta_i(t)x^2 + r_i(t), \tag{5.2}$$

where $\theta_i(t)$ and $r_i(t)$ are real-valued functions, $i = 1, \ldots, N$. Then

$$u_i^e = -[\sigma_i^2(t)\theta_i(t) + N_i^{(i)}(t)]^{-1}B_i(t)\theta_i(t)x, \quad i = 1, \ldots, N. \tag{5.3}$$

Now for relations (b), we have

$$\begin{aligned} x^2\Bigg\{ &\dot\theta_i(t) + 2A(t)\theta_i(t) - 2\theta_i(t)\sum_{j=1}^{N}[\sigma_j^2(t)\theta_j(t) + N_j^{(j)}(t)]^{-1}B_j^2(t)\theta_j(t) + \sigma_0^2(t) \\ &+ \theta_i(t)\sum_{j=1}^{N}\sigma_j^2(t)[\sigma_j^2(t)\theta_j(t) + N_j^{(j)}(t)]^{-2}B_j^2(t)\theta_j^2(t) + \sum_{j=1}^{N}N_j^{(i)}(t)[\sigma_j^2(t)\theta_j(t) \\ &+ N_j^{(j)}(t)]^{-2}B_j^2(t)\theta_j^2(t) + M_i(t)\Bigg\} + \dot r_i(t) + \sigma_{N+1}^2(t)\theta_i(t) = 0, \quad i = 1, \ldots, N, \end{aligned}$$

and we come to the following.

PROPOSITION

Let in the linear-quadratic game (2.1) the functions $A(t)$, $B_i(t)$, $\sigma_j(t)$, $M_i(t)$, $N_j^{(i)}(t)$ be continuous. Let D_i be nonnegative constants, $M_i(t)$ be nonnegative functions and $N_i^{(i)}(t)$ be positive functions for each $t \in [t_0, T]$. Thus

(i) If the set of functions $\theta_i(t)$, $i = 1, \ldots, N$, is the solution of the system of nonlinear differential equations

$$\begin{aligned} &\dot\theta_i(t) + 2A(t)\theta_i(t) - 2\theta_i(t)\sum_{j=1}^{N}[\sigma_j^2(t)\theta_j(t) + N_j^{(j)}(t)]^{-1}B_j^2(t)\theta_j(t) + \sigma_0^2(t) \\ &\quad + \theta_i(t)\sum_{j=1}^{N}\sigma_j^2(t)[\sigma_j^2(t)\theta_j(t) + N_j^{(j)}(t)]^{-2}B_j^2(t)\theta_j^2(t) + M_i(t) + \sum_{j=1}^{N}N_j^{(i)}(t)[\sigma_j^2(t)\theta_j(t) \\ &\quad + N_j^{(j)}(t)]^{-2}B_j^2(t)\theta_j^2(t) = 0, \quad i = 1, \ldots, N, \end{aligned}$$

with boundary conditions

$$\theta_i(T) = D_i, \; i = 1, \ldots, N,$$

and

$$r_i(t) = \int_t^T \sigma_{N+1}^2(\tau)\theta_i(\tau)\,d\tau, \quad i = 1, \ldots, N;$$

(ii) then $W^{(i)}(t, x)$ given by (5.2) are the solutions of the dynamic programming equations (5.1) and u_i^e, $i = 1, \ldots, N$, defined by (5.3) are the Nash-equilibrium strategies for the linear-quadratic game (2.1) considered in this section.

6. CONCLUDING REMARKS

In this paper we consider Nash-equilibrium in stochastic differential games using the approach of Fleming and Rishel[4] to optimal control of stochastic dynamic systems. Thus we come to the solutions of certain partial initial value problems—analogues to the "Bellman equation." This aspect is discussed in [4].

In the linear-quadratic games, considered in Secs. 4 and 5, the existence of the Nash-equilibrium strategies essentially depends on the existence of the solutions of the matrix Riccati and nonlinear differential equations. This problem can be settled using the method of iterations for solving matrix Riccati differential equations or Bellman's method of successive approximations, based on the idea of a quasilinearization of nonlinear differential equations (see [6]).

Nash-equilibrium comes in for criticism mainly as far as the concept of uniqueness is concerned. In Secs. 4 and 5 the existence of the Nash-equilibrium strategies implies their uniqueness, as well.

Finally, we should mention that Nash-equilibrium as a concept of a solution of a stochastic differential game is a firm background for considering other kinds of solutions and their interaction.

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